Preliminary to Ch7 of Hamilton's Ricci flow

- Ric flow $\quad \partial_t g = -2\,Ric \qquad$ (RF)

- normalized $\quad \partial_t g = -2\,Ric + \frac{2}{n} r g \qquad$ (NRF)

- Kulkani – Nomizu product

$$(\omega \circledwedge \eta)_{ijkl} = \omega_{ik}\eta_{jl} + \omega_{jl}\eta_{ik} - \omega_{il}\eta_{jk} - \omega_{jk}\eta_{il}$$

- decomposition of Rm

$$Rm = \frac{R}{2n(n-1)} g^2 + \frac{1}{n-2} \mathring{Ric}\, g + W$$

$$|g^2| = 8n(n-1)$$

Schur's lemma

$n \neq 2$

$$Rm = \frac{R}{2n(n-1)} g^2 \quad \Rightarrow \quad R \text{ const.}$$

pf. $Ric = \frac{1}{n} R g$

2nd Bianchi id $\quad d\,scal = 2\,div(Ric)$

$LHS = dR$

$RHS = 2 g^{ij} \nabla_i (Ric_{jk}) dx^k$

$= 2 g^{ij} g_{jk} \nabla_i \left(\frac{R}{n} \right) dx^k = \frac{2}{n} dR$

if $n > 2 \longrightarrow dR = 0 \Rightarrow R$ const.

Ch 5 Perelman's no collapsing

Einstein - Hilbert

$$E(g) = \int R_g \, d\mu$$

$$\frac{dE}{ds} = \int_M \langle \partial_s g, \tfrac{1}{2} R g - Ric \rangle \, d\mu$$

gradient flow of E. not parabolic

$$\partial_t g = 2 \nabla E(g) = \underset{\sim}{Rg} - 2 Ric$$

want to get rid of this. (comes from $\partial_s d\mu$)
then RHS becomes RHS of Ric flow

To get rid of $\partial_s d\mu$.

- define $\mathcal{E}(g) = \int_M R e^{-f} d\mu$

$$\frac{d\mathcal{E}}{ds} = -\int_M \langle \partial_s g, Ric \rangle e^{-f} d\mu$$

$$+ \int_M \left(-\Delta \operatorname{tr}_g (\partial_s g) + \nabla_i \nabla_j \partial_s g_{ij} \right) e^{-f} d\mu$$

get rid of part of this by considering
$\int_M |\nabla f|^2 e^{-f} d\mu$ w/ $\partial_s (e^{-f} d\mu) = 0$

- Take $\mathcal{F}(g, f) = \int_M \left(R_g + |\nabla f|^2\right) e^{-f} \, d\mu$

$$= \int_M \left(R_g + 2\Delta f - |\nabla f|^2\right) e^{-f} \, d\mu$$

then $\frac{d}{ds} \mathcal{F} = -\int_M \langle \partial_s g, \, Ric + \nabla\nabla f \rangle e^{-f} \, d\mu.$

$\longrightarrow$ gradient flow of $\mathcal{F}$ is

$$\begin{cases} \partial_t g = -2\left(Ric + \nabla\nabla f\right) \\ \partial_t f = -R - \Delta f \end{cases} \qquad (\star)$$

- monotonicity formula

$$\frac{d}{dt} \mathcal{F}(g(t), f(t)) = 2\int_M |Ric + \nabla\nabla f|^2 e^{-f} \, d\mu \geq 0$$

- pullback via diffeo $(\star)$ becomes

$$\begin{cases} \partial_t \tilde{g} = -2Ric_{\tilde{g}} \\ \partial_t \tilde{f} = -R_{\tilde{g}} - \Delta \tilde{f} + |\nabla \tilde{f}|^2 \end{cases}$$

$$\tilde{g} = \varphi^* g \qquad \tilde{f} = \varphi^* f.$$

- same monotonicity formula

$$\frac{d}{dt} \mathcal{F}(g(t), f(t)) = 0 \iff \partial_t g = \mathcal{L}_{\nabla f}\, g$$

Def Perelman's Entropy $\qquad \mathcal{F}$ + scaling

$$\mathcal{W}(g, f, \tau) = \int_M \left(\tau\left(R + |\nabla f|^2\right) + (f - n)\right)(4\pi\tau)^{-n/2} e^{-f} \, d\mu$$

↑ scaling factor $\tau > 0$

Consider
$$\begin{cases} \partial_t g = -2Ric \\ \partial_t f = -\Delta f - R + |\nabla f|^2 + \frac{n}{2\tau} \\ \partial_t \tau = -1 \end{cases}$$

- entropy monotonicity

$$\frac{d}{dt}\mathcal{W} = 2\tau \int_M \left|Ric + \nabla\nabla f - \frac{1}{2\tau} g\right|^2 u \, d\mu \geq 0$$

($u = (4\pi\tau)^{-n/2} e^{-f}$)

Def μ-invariant $\qquad$ monotonicity w.r.t time

$$\mu(g, \tau) = \inf\left\{\mathcal{W} : f \in C^\infty, \int_M (4\pi\tau)^{-n/2} e^{-f} d\mu = 1\right\} > \infty.$$

§ 5.4.3

logarithmic Sobolev ineq. $\Rightarrow$ $\mathcal{W}$ bdd below

Def κ-noncollapsed below the scale ρ

- $\rho \in (0,\infty]$ $\kappa > 0$ if $\forall\ B(x,r),\ r<\rho$
- $|Rm(y)| \leq r^{-2}$ $\forall\ y \in B(x,r)$

$\triangleright$ $$\frac{\operatorname{Vol} B(x,r)}{r^n} \geq \kappa$$

Thm 5.35 (Perelman: no local collapsing)

- $g(t)$, $t \in [0,T)$ RF sol. on closed M^n
- $T < \infty$

$\triangleright$ $\forall\ \rho \in (0,\infty)$, $\exists\ \kappa = \kappa(g(0), T, \rho) > 0$ s.t. $g(t)$ is κ-noncollapsed below the scale ρ for all $t \in [0,T)$.

Remark 5.36. *Perelman's entropy monotonicity formula rules out local collapse for finite time solutions of the Ricci flow on closed manifolds. The idea of the proof is that if a metric g is κ-collapsed at a point p on a distance scale r for κ small and r bounded, then $\mathcal{W}\left(g, f, r^2\right)$ is large and negative, on the order of $\log \kappa$ for f concentrated in a ball of radius r centered at p. This contradicts the monotonicity formula.* of μ

Ch 6. compactness thm Thm 6.35

- $\{(M_i^n, g_i(t), O_i)\}_{i\in\mathbb{N}}$

  O_i: base pt $t=0$

  $t \in (\alpha, \omega)$

complete, pointed sol. of RF s.t.

- $|Rm(g_i(t))|_{g_i(t)} \le C$ on $M_i^n \times (\alpha, \omega)$ for some $C < \infty$.

- $\mathrm{inj}_{g_i(0)}(O_i) \ge \delta > 0$

$\Rightarrow$ $\exists$ subseq. $\xrightarrow{C^k}$ $(M_\infty^n, g_\infty(t), O_\infty)$

$\downarrow$

a completed, pointed sol. of RF

$|Rm(g_\infty)|_{g_\infty} \le C$ on $M_i^n \times (\alpha, \omega)$

pf use Arzela - Ascoli

§ 7.1 Spherical space form

<u>Thm</u> (7.2)

$$- \quad \left| \frac{1}{n-2} \overset{\circ}{Ric} \, g \right|^2 + |W|^2 < \frac{2 \varepsilon_n R^2}{n(n-1)}$$

$$\varepsilon_n = \frac{1}{5}, \ \frac{1}{10}, \ \frac{2}{(n-2)(n+1)} \quad \text{for } n = 4, 5, \geq 6$$

the ineq implies $|Rm|^2 < (4\varepsilon_n + |g^2|^2) \dfrac{R^2}{2n(n-1)}$

▹ unique solution to IVP $\begin{cases} (NRF) & \text{for } t \in [0, \infty) \\ g(0) = g_0 \end{cases}$

▹ $t \to \infty$ $\quad g(t) \to g_\infty$
- converges exp fast in C^k norm
- $scal(g_\infty) \equiv const.$

▹ $M \cong$ spherical space form

Pinching estimate

- $\overset{\circ}{Rm} = Rm - \dfrac{2R}{n(n-1)} Id_{\Lambda^2}$

const sec. curv. Rm

$Id_{\Lambda^2} = \frac{1}{4} g^2.$

estimate how far is g from h:

$sec_h \equiv const > 0$

Prop 7.4

- Pinching $\longrightarrow$ $|\nabla Rm|$ estimate

  - $(M^n, g(t))$ $\quad n \geq 3$.

    (M^n: closed, $scal_{g(t)} > 0$; $g(t)$: Ric flow sol.)

  - $|\mathring{R}m| \leq K R^{1-\varepsilon}$ $\quad K < \infty$, $\varepsilon > 0$

  ▷ $\forall \eta > 0$, $\theta > 0$, $\exists\, C = C(g_0, \eta, \theta) < \infty$ s.t.

  if $R(\bar{x}, \bar{t}) \geq C$, $R(\bar{x}, \bar{t}) \geq \eta \cdot \max_{M^3 \times [0, \bar{t}]} R$

  $\Rightarrow$ $|\nabla Rm|(\bar{x}, \bar{t}) \leq \theta\, R^{3/2}(\bar{x}, \bar{t})$

  Ric scales as g^{-1}, $|\nabla Rm| \sim g^{-3/2}$

pf by contradiction

$|\mathring{Rm}|$ bdd + $R>0$ $\Rightarrow$ $|Rm| < CR$

if Prop false, then $\exists\ \eta>0,\ \theta>0$ s.t. $\forall\ C_i \to \infty$
$\exists\ (x_i, t_i)$ s.t.

$$R(x_i,t_i) \geq \max\left\{ C_i,\ \eta \cdot \max_{M\times[0,t_i]} R \right\}$$

and $|\nabla Rm|(x_i,t_i) \geq \theta\ R^{3/2}(x_i,t_i)$

Perelman no local collapsing (5.41)
+ compactness thm (6.35)

$\longrightarrow\ \exists\ g_i(t)$ dilated sol.

$$\overset{\shortparallel}{R(x_i,t_i)\, g(t_i + R(x_i,t_i)^{-1} t)} \longrightarrow (M^n_\infty, g_\infty)$$ complete ancient sol.

Rm_∞ bdd $\qquad t\in(-\infty,\omega)\quad \omega>0$

$R_\infty > 0$

$|\mathring{Rm}_\infty| = 0$ on $M^n_\infty \times (-\infty,\omega)$

$$\Rightarrow\quad Rm(g_\infty) = \frac{2R_\infty}{n(n-1)}\, Id_{\Lambda^2}.$$

$\hookrightarrow\quad R(g_\infty) \equiv$ const. $\quad \nabla Rm(g_\infty) = 0$

but $\}$ $|\nabla Rm(g_\infty)|(x_\infty,0) > \theta\, R(g_\infty)^{3/2}(x_\infty,0) > 0$